ETMAG
Lecture 5

- Limits of sequences (continued)
- Functions

Theorem (Limits and inequalities) – *started last week*

Suppose $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$ and there exists *k* such that for every *n*>*k*, $a_n \leq b_n$. Then $A \leq B$. (*The limit preserves inequality*).

Proof. (by contradiction). Suppose to the contrary, $A > B$. We put $\varepsilon = \frac{A-B}{2}$. Since $A > B$, $\varepsilon > 0$ as required.

There exists p_a such that for every $n > p_a$, $|a_n - A| < \varepsilon$. This means $-\varepsilon < a_n - A < \varepsilon$. Since $\varepsilon = \frac{A-B}{2}$ we get $a_n > A - \frac{A-B}{2} = \frac{A+B}{2}$.

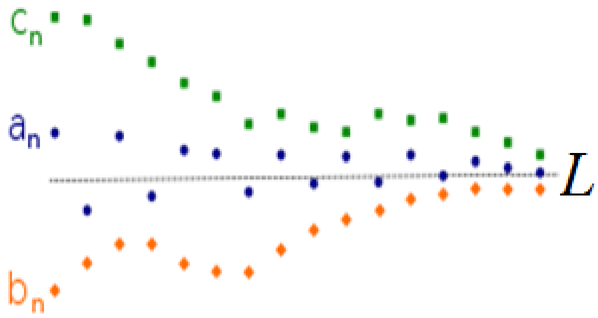
There exists also p_b such that for every $n > p_b$, $|b_n - B| < \varepsilon$ $b_n < \frac{A+B}{2}$. Putting $p = \max(p_a, p_b)$ we have $a_n > \frac{A+B}{2} > b_n$, in short $a_n > b_n$ for every $n > p$. This means that for every $n > \max(k, p)$ we have both $a_n \leq b_n$ and $a_n > b_n$ – a contradiction. QED

Theorem (Sandwich theorem, squeeze lemma)

Consider sequences a_n, b_n and c_n such that: $\lim_{n\to\infty} b_n = L$ and $\lim_{n\to\infty} c_n = L$ and there exists k such that for every $n>k$, $b_n \leq a_n \leq c_n$. Then the sequence a_n is convergent and $\lim_{n\to\infty} a_n = L$.

Proof. For every $\varepsilon > 0$ we can choose p (good for both sequences c_n and b_n) such that for every $n > p$, $|c_n - L| < \varepsilon$ and $|b_n - L| < \varepsilon$. This guarantees that for every $n > p$

$$L - \varepsilon < b_n \leq a_n \leq c_n < L + \varepsilon, \text{ i.e., } |a_n - L| < \varepsilon. \text{ QED}$$



(the graph from Wikipedia)

This theorem is incredibly useful. There are tons of sequences whose limits cannot be calculated by arithmetic operations on known, elementary limits and with the squeeze theorem they become … trivial.

Example.

Find $\lim_{n\to\infty} a_n$ where $a_n = \frac{\sin n}{n}$.

$\lim_{n\to\infty} \sin n$ does not exists (which is not quite trivial) but

for each n, $\frac{-1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$ and $\lim_{n\to\infty} \frac{-1}{n} = 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$.

Hence, by squeeze lemma, $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

Common pitfalls.

A student is instructed to check if a sequence x_n is convergent. They remember vaguely "*Uncle Tom said something about hamburgers*". They find some sequences z_n and y_n. Now, it may go several ways:

- z_n and y_n converge to the same limit so they announce proudly that x_n is convergent. But they never bother to check if $(\forall n)\ z_n \leq x_n \leq y_n$. LOL, score 0.
- $(\forall n)\ z_n \leq x_n \leq y_n$ but y_n or z_n is divergent. Whatever their conclusion, it makes no sense, LOL.
- $(\forall n)\ z_n \leq x_n \leq y_n$, z_n and y_n converge but to different limits. Just as the last one.

I've seen those many times and I don't want to see them again. Or else …

Theorem.

Every convergent sequence is bounded.

Comprehension. Prove the theorem by first principles (it means directly from the definition of the limit).

Fact.

Not every bounded sequence is convergent.

Comprehension. Find a divergent bounded sequence.

Theorem.

Every bounded and monotonic sequence is convergent.

Proof outline.

In the case of a nondecreasing bounded sequence (a_n) we prove that $L = \sup\{a_n : n \in \mathbb{N}\}$ is the limit of a_n. L exists because every bounded set of real numbers has the least upper bound. The trick (not very difficult) is to show that L is the limit for a_n.

In the same spirit, if (a_n) is nonincreasing we take $L = \inf\{a_n : n \in \mathbb{N}\}$.

Theorem. (Euler)

The sequence $(1+\frac{1}{n})^n$ is convergent.

Hint. It turns out that the sequence is increasing and bounded from above, hence convergent by the last theorem.

Definition.

The limit of the sequence $(1+\frac{1}{n})^n$ is denoted by e and is called the *Euler number*. Its approximation is e=2.7182818284590452…

Definition

We say that a sequence a_n *diverges to* ∞ iff

$(\forall r \in \mathbb{R})(\exists k \epsilon \mathbb{N})(\forall n > k)\ a_n > r$

We denote this by $\lim\limits_{n\to\infty} a_n = \infty$

In a similar way we define *divergence to* $-\infty$:

$(\forall r \in \mathbb{R})(\exists k \epsilon \mathbb{N})(\forall n > k)\ a_n < r$.

So, in total, a sequence may be *convergent* (to a number), *divergent* or *divergent to* (plus or minus infinity). Note that $+\infty$ and $-\infty$ are not numbers.

Theorem (Properties of infinite limits)

- If $\lim_{n\to\infty} a_n = \pm\infty$ then $\lim_{n\to\infty} \frac{1}{a_n} = 0$ (*vulgar and misleading form*: $\frac{1}{\infty} = 0$)
- If $\lim_{n\to\infty} a_n = \infty$ and (b_n) is bounded from below then $\lim_{n\to\infty} (a_n + b_n) = \infty$ ($\infty + c = \infty$)
- If $\lim_{n\to\infty} a_n = \infty$ and for every n $b_n \geq c$ for some $c>0$, then $\lim_{n\to\infty} a_n b_n = \infty$ ($c\infty = \infty$)
- If $\lim_{n\to\infty} a_n = \infty$ and $a_n \leq b_n$ for every n then $\lim_{n\to\infty} b_n = \infty$ (*vm form* squeeze lemma for infinities)

Theorem

Important limits to remember:

- If a>1 then $\lim_{n\to\infty} a^n = \infty$
- If $|a|$<1 then $\lim_{n\to\infty} a^n = 0$
- If a>0 then $\lim_{n\to\infty} \sqrt[n]{a} = 1$
- $\lim_{n\to\infty} \sqrt[n]{n} = 1$
- $\lim_{n\to\infty} (1 + \frac{1}{n})^n = e$
- $\lim_{n\to\infty} n \sin\frac{1}{n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1$

FUNCTIONS

We consider functions $f: X \to Y$ where X and Y are subsets of $\mathbb{R}$. Given a formula defining *f*, the largest subset X of $\mathbb{R}$ on which the formula makes sense is called the *natural domain* of *f*.

It is often convenient to choose Y = *f*(X). Then *f* is a surjection (*onto*) and Y is called the *set of values* for *f*.

For example

$\sin : \mathbb{R} \to [-1; 1]$, (we put $Y = [-1; 1]$)

$\ln: (0; \infty) \to \mathbb{R}$ (we put $X = (0; \infty)$)

$\tan : \mathbb{R} \setminus \{k\pi + \frac{\pi}{2} | k \in \mathbb{Z}\} \to \mathbb{R}$

Definition.

A function is called *constant on* A, $A \subseteq X$, iff

$$(\exists c \in \mathbb{R})(\forall x \in A) f(x) = c$$

Definition.

Let $A \subseteq X$. A function $f: X \to Y$ is called

- *increasing on* A iff$(\forall x, y \in A)[x < y \Rightarrow f(x) < f(y)]$
- *nondecreasing on* A iff $(\forall x, y \in A)[x < y \Rightarrow f(x) \leq f(y)]$
- *decreasing on* A iff $(\forall x, y \in A)[x < y \Rightarrow f(x) > f(y)$
- *nonincreasing on* A iff $(\forall x, y \in A)[x < y \Rightarrow f(x) \geq f(y)]$

Definition.

A function f is said to be *even* iff $\left(\forall x \in Dom(f)\right) f(-x) = f(x)$

A function f is said to be *odd* iff $\left(\forall x \in Dom(f)\right) f(-x) = -f(x)$

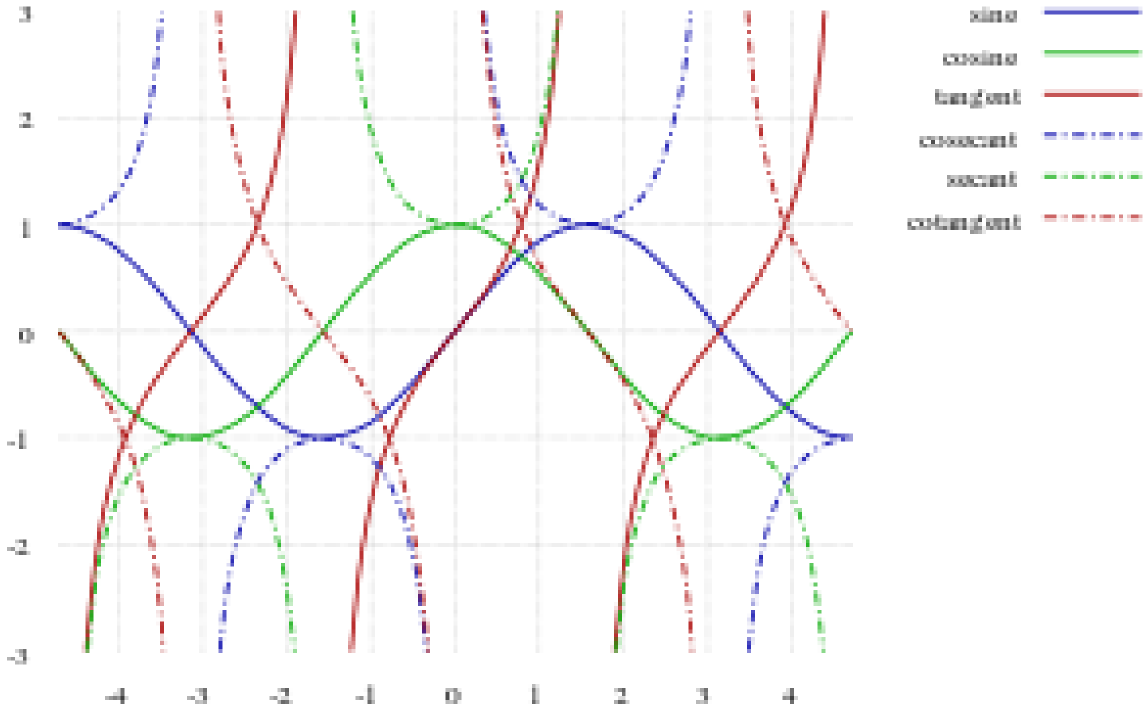
Example.

Sinus is increasing on every closed interval of the form $[2k\pi - \frac{\pi}{2}; 2k\pi + \frac{\pi}{2}]$ and decreasing on every closed interval of the form $[2k\pi + \frac{\pi}{2}; 2k\pi - \frac{\pi}{2}]$, where k is an integer.

The *floor* function, $\lfloor x \rfloor$ = the largest integer l such that $l \leq x$ is globally nondecreasing. It is constant on every interval of the form $[k;k+1)$.

Example.

Consider *tan* x. The domain (or natural domain), of *tan* x is $\mathbb{R} \setminus \{k\pi+\frac{\pi}{2} \,|\, k\in\mathbb{Z}\}$. *tan x* is increasing on every interval (a,b) which is contained in its domain, but it is not increasing *globally.* (graph from Wikipedia).



Comprehension.

What can you say about a set A and a function f if

- f is at the same time nonincreasing and nondecreasing on A
- f is at the same time increasing and decreasing on A

Definition. (*Reminder*)

Given a function $f : X \to Y$, if there exists a function $g : Y \to X$ such that $f \circ g = id_Y$ and $g \circ f = id_X$ then g is called the *inverse function* for f or f-*inverse* and is denoted by f^{-1}.

Fact.

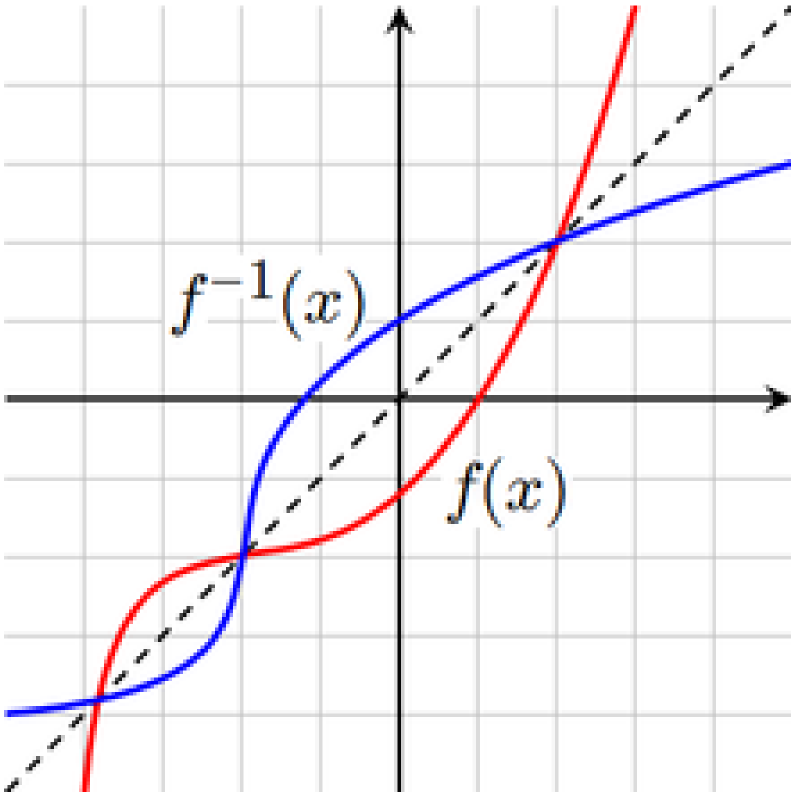
- f is the inverse for g iff g is the inverse for f.
- f is the inverse for g iff for every $x \in X$ and $y \in Y$

$$f(x)=y \Leftrightarrow g(y)=x$$

- f is invertible iff f is "one-to-one" and "onto".

Fact.

The graph of f^{-1} is the mirror reflection of the graph of f in the line $y = x$. (Image from Wikipedia.)



ELEMENTARY FUNCTIONS

Definition.

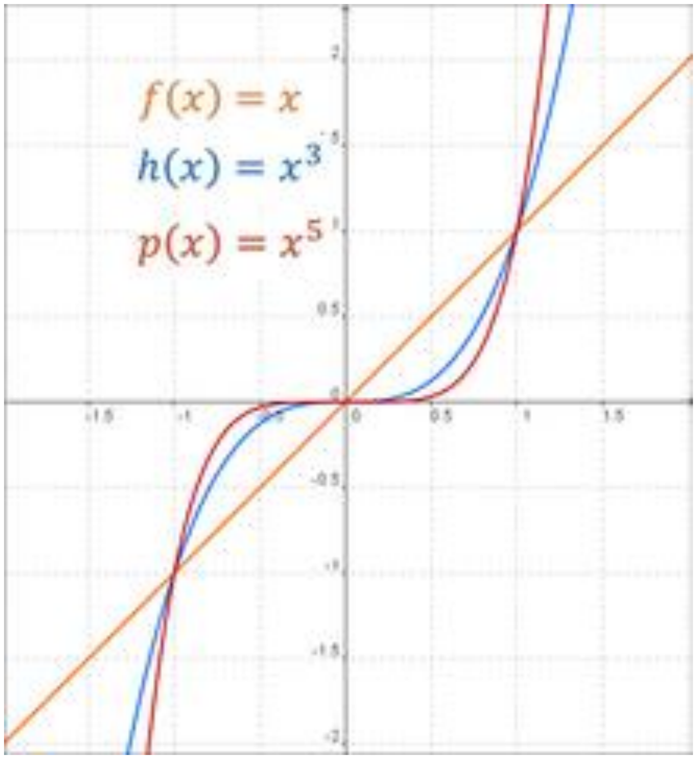
The set of *elementary functions* consists of:

- constant functions (constant on $\mathbb{R}$)
- *id* function (*id*(x) = x, the "do nothing" function
- trigonometric functions
- power functions (x^b where b is a real number, not necessarily an integer)
- exponential functions (functions of the form $f(x) = a^x$)
- functions obtained by arithmetic operations on elementary functions (sums, product, quotients …)
- compositions of elementary functions
- inverses of elementary functions

Fact.

- Applying only addition and multiplication to constant functions and the identity function we get all polynomials.
- Applying division to polynomials we get rational functions.

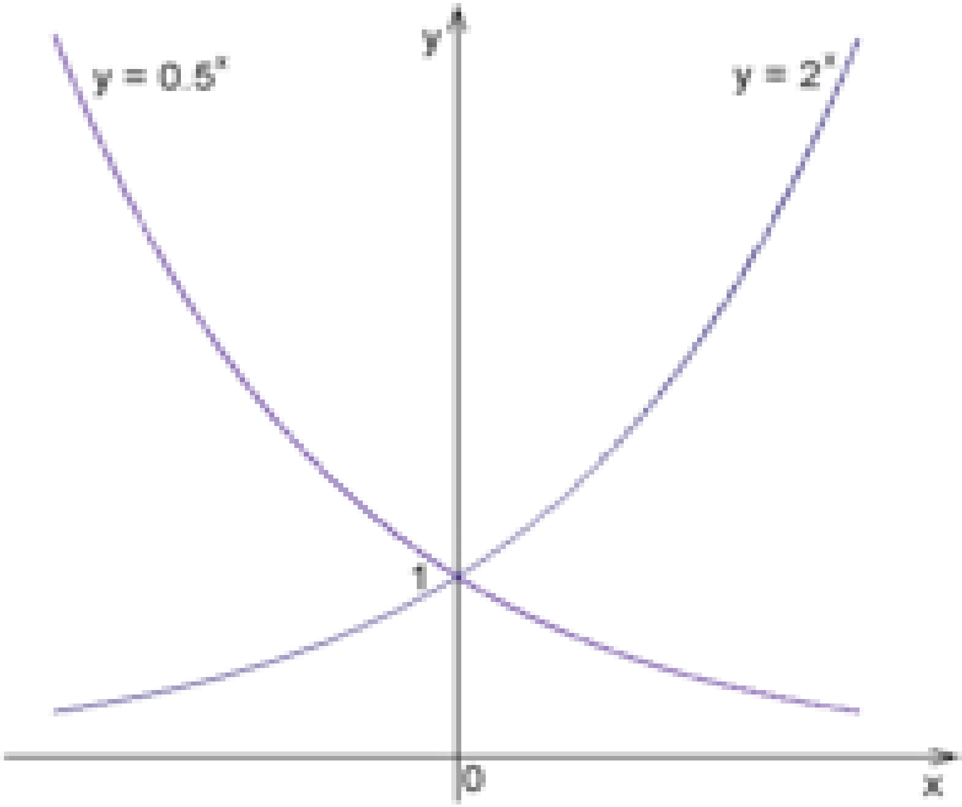
Power functions are functions of the form $f(x) = x^a$ where a is a constant. If a=1, x^a becomes the identity function.



The image from Wikipedia

Exponential functions are functions of the form $f(x) = a^x$ where a is a positive constant different from 1.

If $a > 1$ then a^x is increasing, otherwise it is decreasing.



The image from Wikipedia

Exponential functions **should not be confused with** *power functions*.

Exponential functions have a constant base, the variable is in the exponent, as in 2^x.

In power functions the variable is in the base, the exponent is constant, as in x^2.

Fact. (Properties of powers)

- $a^b a^c = a^{b+c}$, this implies a^0=1 and $a^{-b} = \frac{1}{a^b}$
- $\left(a^b\right)^c = a^{bc}$
- $a^b c^b$=$(ac)^b$